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CERTAIN GENERAL PROPERTIES OF FUNCTIONS.*

BY HENRY BLUMBERG.

Introduction. Definitions and statement of results. Let $g(x)$ be a real, one-valued, bounded, continuous or discontinuous function defined in the interval (a, b) . The functional values of $g(x)$ in the subinterval (α, β) of (a, b) have a least upper-bound, a greatest lower-bound and a saltus, which we denote respectively by

$$u(g, \alpha\beta), l(g, \alpha\beta) \text{ and } s(g, \alpha\beta) = u(g, \alpha\beta) - l(g, \alpha\beta).$$

The *upper-bound*, the *lower-bound*† and the *saltus* of $g(x)$ at the fixed point x of (a, b) are defined respectively as the (greatest) lower-bound of $u(g, \alpha\beta)$, the (least) upper-bound of $l(g, \alpha\beta)$ and the (greatest) lower-bound of $s(g, \alpha\beta)$ for all possible subintervals (α, β) of (a, b) that contain x as interior point.‡ With the given function $g(x)$, there are thus associated three new functions of x , the *upper-bound function*, the *lower-bound function* and the *saltus function*, which we denote by

$$u(g, x), l(g, x) \text{ and } s(g, x)$$

respectively.§ It is seen that

$$s(g, x) \equiv u(g, x) - l(g, x).$$

Let us, for the sake of brevity, write

$$s(g, x) \equiv s'(x), \quad s(s', x) \equiv s''(x), \quad s(s'', x) \equiv s'''(x), \quad \dots$$

Sierpiński|| has shown that

$$s''(x) \equiv s'''(x).$$

* Presented to the American Mathematical Society, April, 1914, and December, 1916.

† Throughout the paper, we use the expressions, "the upper-bound" and "the lower-bound," in the sense of "the least upper-bound" and "the greatest lower-bound," respectively.

‡ Of course, it will be understood, that in case $x = a$ or b —and only then—we permit α or β to coincide with x . This remark applies also to similar situations below.

§ The functions $u(g, x)$ and $l(g, x)$ are often, though not quite unobjectionably, called the "maximum" and the "minimum" functions belonging to $g(x)$. Cf., for example, Hobson, *The Theory of Functions of a Real Variable* (1907), art. 180.

|| *Bulletin de l'Académie des Sciences de Cracovie* (1910), pp. 633-634.

The chief aim of the present paper is to communicate a number of companion propositions to Sierpinski's theorem. The new results are based on the definition of other types of saltus, which immediately suggest themselves and arise from the one described above, when certain specified subsets of the range of the independent variable may be neglected.

The first new type arises when *finite* subsets may be neglected. For every subinterval (α, β) of (a, b) , there evidently exists a number, which we denote by $u_f(g, \alpha\beta)$, uniquely characterized by the following double property:* For every $\epsilon > 0$, on the one hand, the set of points of the interval (α, β) for which $g(x) > u_f(g, \alpha\beta) + \epsilon$ is finite, while on the other hand, there exists an infinite set of points of (α, β) at which $g(x) > u_f(g, \alpha\beta) - \epsilon$. This number $u_f(g, \alpha\beta)$ is the lower-bound of all possible upper-bounds that $g(x)$ may have in the interval (α, β) , in case a finite set of points may be neglected. Likewise, there is a number, which we denote by $l_f(g, \alpha\beta)$, characterized by the property that for every $\epsilon > 0$, the set of points of (α, β) where $g(x) < l_f(g, \alpha\beta) - \epsilon$ is finite, whereas, the set of points of (α, β) where $g(x) < l_f(g, \alpha\beta) + \epsilon$ is infinite. $l_f(g, \alpha\beta)$ is the upper-bound of all possible lower-bounds of $g(x)$ in (α, β) , when a finite number of points may be neglected. Finally, we denote by $s_f(g, \alpha\beta)$ the lower-bound of the saltus of $g(x)$ in (α, β) , in case a finite number of points may be neglected. Evidently

$$s_f(g, \alpha\beta) = u_f(g, \alpha\beta) - l_f(g, \alpha\beta).$$

We shall designate the numbers

$$u_f(g, \alpha\beta), l_f(g, \alpha\beta) \text{ and } s_f(g, \alpha\beta)$$

as “the *f*-upper-bound,” “the *f*-lower-bound” and “the *f*-saltus” of $g(x)$ in the interval (α, β) . As in the case where no point may be neglected, we now define “the *f*-upper-bound,” “the *f*-lower-bound” and “the *f*-saltus” of $g(x)$ at the fixed point x of (a, b) , as the lower-bound of $u_f(g, \alpha\beta)$, the upper-bound of $l_f(g, \alpha\beta)$ and the lower-bound of $s_f(g, \alpha\beta)$ for all possible subintervals (α, β) of (a, b) that contain x as interior point. With the given function $g(x)$, we have thus associated “the *f*-upper-bound function,” “the *f*-lower-bound function” and “the *f*-saltus function,” which we denote by

$$u_f(g, x), l_f(g, x) \text{ and } s_f(g, x)$$

respectively.

In the second place, the subsets that may be neglected are *denumerable*. As before, there exists a number, which we denote by $u_d(g, \alpha\beta)$, uniquely characterized by the property that, for every $\epsilon > 0$, the set of points of

* Cf. Baire, Acta Mathematica, vol. 30 (1906), pp. 21 and 22.

(α, β) where $g(x) > u_d(g, \alpha\beta) + \epsilon$ is denumerable, whereas the set of points where $g(x) > u_d(g, \alpha\beta) - \epsilon$ is non-denumerable; this number $u_d(g, \alpha\beta)$ we call "*the d-upper-bound*" of $g(x)$ in (α, β) .* Precisely as before, we define the related numbers

$$l_d(g, \alpha\beta), s_d(g, \alpha\beta), u_d(g, x), l_d(g, x)$$

and "*the d-saltus function*" $s_d(g, x)$.

In the third place, "*exhaustible*" sets (i. e., sets of first category†) may be neglected. We then obtain "*the e-saltus function*" $s_e(g, x)$, together with the related numbers.

Finally, sets of (Lebesgue) *zero measure* may be neglected. We then obtain "*the z-saltus function*" $s_z(g, x)$, together with the related functions.

As in the case where no point may be neglected, we write

$$s_f(g, x) \equiv s_f'(x), s_f(s_f', x) \equiv s_f''(x), s_f(s_f'', x) \equiv s_f'''(x), \dots;$$

and similarly for the *d*-saltus, the *e*-saltus and the *z*-saltus functions. Also

$$s_f'(x) = u_f(g, x) - l_f(g, x),$$

and similarly for the other saltus functions.

Having defined the new types of saltus we had in view, we may now state the corresponding analogues of Sierpiński's theorem. The most interesting and least obvious results are that

$$s_d'''(x) \equiv s_d^{\text{IV}}(x)$$

and

$$s_z'''(x) \equiv s_z^{\text{IV}}(x);$$

moreover, as examples show, $s_d''(x)$ and $s_z''(x)$ may be different from $s_d'''(x)$ and $s_z'''(x)$ respectively. In the case of the *f*-saltus, the functions $s_f^{(n)}(x)$ ($n = 1, 2, \dots$) may all be different. In the case of the *e*-saltus, it follows easily enough that $s_e'''(x)$ is always identically zero, whereas $s_e''(x)$ may be different from $s_e'''(x)$.

Because of our negative result in the case of the *f*-saltus, we are naturally led to define $s_f^{(\beta)}(x)$ for transfinite β 's. Our result will show that it is sufficient to confine ourselves to transfinite ordinal numbers belonging to Cantor's second class. If β is not a limiting number, $\beta - 1$ exists, and we define $s_f^{(\beta)}(x)$ as equal to $s_f(s_f^{(\beta-1)}, x)$. For our purpose, therefore, all we have to do now is to define $s_f^{(\beta)}(x)$, in case β is a limiting number, in terms of the functions $s_f^{(\nu)}(x)$, where $\nu < \beta$. This we do simply by means of the equation

* It follows that the set of points of (α, β) for which $g(x) > u_d(g, \alpha\beta)$ is also denumerable. This fact may be used to modify our definition of the *d*-upper-bound. An analogous remark applies to the types of saltus defined below, but not, however, to the *f*-saltus.

† Cf. Denjoy, Journal de Mathématiques, ser. 7, vol. 1 (1915), pp. 122-125.

$$s_f^{(\beta)}(x) = \lim_{\nu_n \rightarrow \beta} s_f^{(\nu_n)}(x),$$

where $\{\nu_n\}$ is a sequence of numbers less than β . It is easily seen, especially in connection with our later discussion, that this limit always exists and is independent of the particularly chosen sequence $\{\nu_n\}$.

Our positive result for the case of the f -saltus may now be stated as follows:

There exists a number β of the first or the second class, such that

$$s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x).$$

Furthermore, it is shown by means of an example, that if β is a given number of the first or the second class, then the functions $s_f^{(\nu)}(x)$ ($1 \leq \nu \leq \beta$) may all be different, whereas $s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x)$.

Attention is also directed to the relation

$$s_d(s_f'', x) \equiv s_f^{(\beta)}(x)$$

of § 4 connecting the f -saltus and the d -saltus.

For the sake of completeness, we include in our discussion the proof of Sierpiński's Theorem (§ 2). § 1 gives a lemma due to Baire, which simplifies later proofs. §§ 3, 4, 5, and 6 treat the d -saltus, the f -saltus, the e -saltus and the z -saltus respectively. The Appendix indicates generalizations.

1. A lemma due to Baire.* Suppose that with each subinterval (α, β) of (a, b) there is associated a real number $\varphi_{\alpha\beta}$ in such a way that if (γ, δ) is contained in (α, β) we have $\varphi_{\gamma\delta} \leq \varphi_{\alpha\beta}$; moreover, $\varphi_{\alpha\beta}$ has a finite lower-bound. Let $\varphi(x)$ be defined as the lower-bound of $\varphi_{\alpha\beta}$ for all possible intervals (α, β) containing x in their interior.† Then $\varphi(x)$ is an upper-semi-continuous function.

That $\varphi(x)$ exists follows from the finiteness of the lower-bound of $\varphi_{\alpha\beta}$. Moreover, $\epsilon > 0$ having been given in advance, a subinterval (α, β) exists containing the fixed point x in its interior, such that $\varphi_{\alpha\beta} < \varphi(x) + \epsilon$. From our definition of $\varphi(x)$, we have for every point x_1 of (α, β) :

$$\varphi(x_1) \leq \varphi_{\alpha\beta} < \varphi(x) + \epsilon.$$

Hence $\varphi(x)$ is upper-semi-continuous at x .

By making $\varphi_{\alpha\beta}$ successively equal to $u(g, \alpha\beta)$, $u_f(g, \alpha\beta)$, $u_d(g, \alpha\beta)$, $u_e(g, \alpha\beta)$ and $u_z(g, \alpha\beta)$, we see that

$$u(g, x), u_f(g, x), u_d(g, x), u_e(g, x) \text{ and } u_z(g, x)$$

* Cf. Acta Mathematica, I. c., p. 16. Baire's situation is more general, but see Appendix.

† For the case where x is a or b , compare footnote above.

are upper-semi-continuous functions. Similarly, the functions $-l(g, x)$, $-l_f(g, x)$, etc., are upper-semi-continuous. Since the sum of two upper-semi-continuous functions is again upper-semi-continuous, it follows, in virtue of

$$s'(x) = u(g, x) - l(g, x), \quad s_f'(x) = u_f(g, x) - l_f(g, x),$$

etc., that

$$s'(x), s_f'(x), s_d'(x), s_e'(x), \text{ and } s_z'(x)$$

are upper-semi-continuous functions.

2. The ordinary saltus. THEOREM (Sierpiński).

$$s''(x) \equiv s'''(x) \equiv s^{\text{IV}}(x) \equiv \dots$$

Proof. According to § 1, $s'(x)$ is upper-semi-continuous and therefore continuous at every point of an everywhere dense subset $\mathfrak{S}$ of (a, b) .* Since the saltus of a function is zero at its points of continuity, it follows that $s''(x)$, the saltus function belonging to $s'(x)$, is zero everywhere in $\mathfrak{S}$. Moreover, according to § 1, $s''(x)$, being a saltus function, is upper-semi-continuous and furthermore, everywhere ≥ 0 . From the upper-semi-continuity of $s''(x)$, it follows that $u(s'', x)$, the upper-bound function belonging to $s''(x)$, is equal to $s''(x)$. From the fact that $s''(x) \geq 0$ everywhere and $= 0$ at the points of $\mathfrak{S}$, it follows that $l(s'', x)$, the lower-bound function belonging to $s''(x)$, is identically zero. Therefore,

$$s'''(x) \equiv s(s'', x) \equiv u(s'', x) - l(s'', x) \equiv s''(x).$$

By taking $g(x)$ successively equal to $s'(x)$, $s''(x)$, $\dots$, we see that

$$s'''(x) \equiv s^{\text{IV}}(x) \equiv s^{\text{V}}(x) \equiv \dots$$

3. The d -saltus. THEOREM. $s_d'''(x) \equiv s_d^{\text{IV}}(x) \equiv s_d^{\text{V}}(x) \equiv \dots$

The following example shows that $s_d''(x)$ need not equal $s_d'''(x)$. Let $(a, b) \equiv (0, 1)$; $g(x) = 0$, if x satisfies the inequalities $0 \leq x \leq \frac{1}{2}$, and is such that at most a finite number of 1's are present in its decimal development; $g(x) = 1$ at all other points of $(0, 1)$. It is seen that every subinterval of $(0, \frac{1}{2})$ has a non-denumerable number of points where $g(x) = 0$, and also a non-denumerable set of points where $g(x) = 1$. Hence,

$$s_d'(x) = 1 \quad (0 \leq x \leq \tfrac{1}{2}), \quad s_d'(x) = 0 \quad (\tfrac{1}{2} < x \leq 1);$$

$$s_d''(x) = 0 \quad (x \neq \tfrac{1}{2}), \quad s_d''(\tfrac{1}{2}) = 1; \quad s_d'''(x) \equiv 0.$$

(This example may be modified so that $s_d''(x) \neq s_d'''(x)$ and $s_d'''(x) \neq 0$.)

We now proceed to the proof of the theorem. According to § 1,

* Cf., for example, Hobson, l. c., pp. 527 and 525.

$s_d'(x)$ is upper-semi-continuous and therefore continuous at every point of a set $\mathfrak{S}$, which contains in every subinterval of (a, b) a non-denumerable set of points.* At every point of continuity of $s_d'(x)$, we have $s(s_d', x) = 0$. Since it follows obviously from our definitions that

$$0 \leq s_d(s_d', x) \leq s(s_d', x),$$

we have $s_d(s_d', x) = s_d''(x) = 0$ at every point of $\mathfrak{S}$. This shows, in virtue of the non-negativity of $s_d''(x)$, that

$$l_d(s_d'', x) \equiv 0.$$

Moreover, since, according to § 1, $s_d''(x)$ is an upper-semi-continuous function,

$$u_d(s_d'', x) \leq u(s_d'', x) \leq s_d''(x).$$

Hence,

$$s_d'''(x) = u_d(s_d'', x) - l_d(s_d'', x) = u_d(s_d'', x) \leq s_d''(x).$$

Now let $\mathfrak{R}''$ be the planar set of points whose coördinates are

$$x = x, \quad y = s_d''(x) \quad (a \leq x \leq b).$$

If P is any point of the plane, *whether belonging to $\mathfrak{R}''$ or not*, then either there exists a neighborhood of P containing at most a denumerable subset of $\mathfrak{R}''$, or every neighborhood of P contains a non-denumerable subset of $\mathfrak{R}''$. In the first case, we call P a “*point of denumerability*” of $\mathfrak{R}''$; in the second case, a “*point of non-denumerability*” of $\mathfrak{R}''$.† We have just seen that $s_d'''(x) \leq s_d''(x)$ everywhere. We show furthermore that *for every ξ of (a, b) , the point $(\xi, s_d'''(\xi))$ is the highest point of non-denumerability of $\mathfrak{R}''$ on the line $x = \xi$* .‡ For suppose first that (ξ, η) is a point of denumerability of $\mathfrak{R}''$ such that every point in the line $x = \xi$ above (ξ, η) is also a point of denumerability of $\mathfrak{R}''$. We must then show that $s_d'''(\xi) < \eta$. Every point (ξ, y) , $\eta \leq y \leq u(g, ab)$, being a point of denumerability of $\mathfrak{R}''$, may be enclosed in a rectangle with sides parallel to the coördinate axes containing in its interior and on its boundary at most a denumerable subset of $\mathfrak{R}''$. According to the Borel theorem on interval sets, no more than a finite number of distinct rectangles need to be used. It is then seen that, by taking δ sufficiently small, we may cut out from the region covered by these rectangles a region,

$$\xi - \delta \leq x \leq \xi + \delta, \quad \eta - \delta \leq y \leq u(g, ab),$$

* See Hobson, I. c., pp. 527, 525, and 244.

† A “point of non-denumerability” is more usually called a “condensation point.”

‡ The existence of such a point follows from the fact that the set of points of non-denumerability of $\mathfrak{R}''$ is closed. For our later purposes, it is sufficient to show merely that $(\xi, s_d'''(\xi))$ is a point of non-denumerability of $\mathfrak{R}''$.

containing in its interior and on its boundary at most a denumerable set of points. Therefore, the function $s_d''(x)$ has everywhere in the interval $(\xi - \delta, \xi + \delta)$ a value less than $\eta - \delta$ except possibly at the points of a denumerable set. Therefore, $u_d(s_d'', \xi) \leq \eta - \delta$ and

$$s_d'''(\xi) = u_d(s_d'', \xi) \leq \eta - \delta < \eta.$$

Secondly, suppose that (ξ, η) is a point of non-denumerability of $\mathfrak{R}''$. Then it is clear that $u_d(s_d'', \xi) \geq \eta$ and

$$s_d'''(\xi) = u_d(s_d'', \xi) \geq \eta.$$

This result, in conjunction with the inequality $s_d'''(\xi) < \eta$ in the previous case, shows that $(\xi, s_d'''(\xi))$ is actually the highest point of non-denumerability of $\mathfrak{R}''$ on the line $x = \xi$.

If, in particular, $(\xi, s_d'''(\xi))$ is itself a point of non-denumerability of $\mathfrak{R}''$, it follows that $s_d'''(\xi) \geq s_d''(\xi)$. In virtue of the inequality $s_d'''(\xi) \leq s_d''(\xi)$ obtained above, it thus follows that

$$s_d'''(\xi) = s_d''(\xi)$$

for a point $(\xi, s_d''(\xi))$ of non-denumerability of $\mathfrak{R}''$.

According to a known theorem, the points of $\mathfrak{R}''$ that are points of denumerability of $\mathfrak{R}''$ constitute at most a denumerable set. Hence, if we designate by $\mathfrak{R}'''$ the set of points $(x, s_d'''(x))$, $a \leq x \leq b$, it follows from the fact that $s_d'''(\xi) = s_d''(\xi)$ when $(\xi, s_d''(\xi))$ is a point of non-denumerability of $\mathfrak{R}''$, that $\mathfrak{R}''$ and $\mathfrak{R}'''$ coincide except possibly at a denumerable set of points; and furthermore, that every point of non-denumerability of $\mathfrak{R}''$ is likewise a point of non-denumerability of $\mathfrak{R}'''$. But precisely as above, it is seen by use of $\mathfrak{R}'''$ instead of $\mathfrak{R}''$, that

$$s_d^{\text{IV}}(\xi) = s_d'''(\xi)$$

whenever $(\xi, s_d'''(\xi))$ is a point of non-denumerability of $\mathfrak{R}'''$. Since, however, according to the above, every point $(\xi, s_d'''(\xi))$ is a point of non-denumerability of $\mathfrak{R}''$, and therefore of $\mathfrak{R}'''$, it follows that for every x in the interval (a, b) the equation $s_d^{\text{IV}}(x) = s_d'''(x)$ holds.

As in the case of Sierpiński's theorem,

$$s_d^{\text{IV}}(x) \equiv s_d^{\text{V}}(x) \equiv \dots$$

Remark. If $s_{\aleph'}(x)$, $s_{\aleph''}(x)$, $s_{\aleph'''}(x)$, $\dots$ represent the successive saltus functions that arise when, instead of neglecting denumerable sets (i. e., sets of cardinal number $\aleph_0$), we neglect sets of cardinal number $\aleph$, where $\aleph$ is any cardinal number $< c$, the cardinal number of the continuum, then $s_{\aleph'''}(x) \equiv s_{\aleph^{\text{IV}}}(x) \equiv \dots$. The proof of this more general proposition, being entirely analogous to that of the special case $\aleph = \aleph_0$, is omitted.

4. The f -saltus. The following example shows that if β is an ordinal number of the first or the second class, there exist functions $g(x)$, such that the functions $s_f^{(\nu)}(x)$ are all different for $1 \leq \nu \leq \beta$, whereas $s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x) \equiv \dots$. For let $\mathfrak{S}$ be a reducible set contained in the interval (a, b) such that $\mathfrak{S}^{(\beta)}$, the β th derivative of $\mathfrak{S}$, is the null-set, whereas every derivative $\mathfrak{S}^{(\nu)}$, $\nu < \beta$, contains points.* Let $g(x) = 1$ or 0 according as x belongs or does not belong to $\mathfrak{S}$. Clearly $s_f'(x) = 1$ or 0 , according as x belongs or does not belong to $\mathfrak{S}'$ (the first derivative of $\mathfrak{S}$); $s_f''(x) = 1$ or 0 according as x belongs or does not belong to $\mathfrak{S}''$; and so on, till we reach $\mathfrak{S}^{(\beta)}$, no matter whether β belongs to the first or to the second class of numbers (cf. the definition in the Introduction of $s_f^{(\nu)}(x)$ for transfinite ν 's). $g(x)$ is thus seen to have the required property.

By a slight modification of this example, we could retain the desired result, and, in addition, make $s_f^{(\beta)}(x) \neq 0$.

We now prove the

THEOREM. *For every given function $g(x)$,† there exists a number β of the first or the second class, such that*

$$s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x) \equiv s_f^{(\beta+2)}(x) \equiv \dots$$

Proof. As in the case of the d -saltus, we see that $s_f''(x)$ is a nowhere-negative, upper-semi-continuous function, which is 0 in an everywhere dense set of points of (a, b) . Let $\mathfrak{R}''$ denote the planar set of points whose coördinates are

$$x = x, \quad y = s_f''(x) \quad (a \leq x \leq b);$$

more generally, $\mathfrak{R}^{(\beta)}$ ($\beta =$ any number of the first or the second class), the planar set of points whose coördinates are

$$x = x, \quad y = s_f^{(\beta)}(x) \quad (a \leq x \leq b).$$

Furthermore, let $\mathfrak{R}_d^{(\beta)}$ represent the set of points of $\mathfrak{R}^{(\beta)}$ that are points of denumerability of $\mathfrak{R}^{(\beta)}$; and $\mathfrak{R}_{nd}^{(\beta)}$, the points of $\mathfrak{R}^{(\beta)}$ that are points of non-denumerability of $\mathfrak{R}^{(\beta)}$. If $(\xi, s_f''(\xi))$ belongs to $\mathfrak{R}_{nd}''$, it follows, in virtue of the upper-semi-continuity of $s_f''(x)$, that $u_f(s_f'', \xi) = s_f''(\xi)$. Moreover, because $s_f''(x) \geq 0$, and the zeros of $s_f''(x)$ are everywhere dense in (a, b) , we have $l_f(s_f'', x) \equiv 0$. Therefore,

$$s_f'''(\xi) = u_f(s_f'', \xi) - l_f(s_f'', \xi) = s_f''(\xi)$$

if $(\xi, s_f''(\xi))$ belongs to $\mathfrak{R}_{nd}''$. This shows that every point of $\mathfrak{R}_{nd}''$ is also a point of $\mathfrak{R}'''$. But furthermore, every point of $\mathfrak{R}_{nd}''$ is also a point of

* The existence of such a set $\mathfrak{S}$ follows easily by means of finite and transfinite induction.

† Satisfying, of course, the conditions of uniqueness and boundedness demanded throughout.

$\mathfrak{R}_{nd}'''$. For according to a known theorem, $\mathfrak{R}_d''$ is at most a denumerable set. Hence $\mathfrak{R}'''$ and $\mathfrak{R}''$, coinciding in the points of $\mathfrak{R}_{nd}''$, differ at most in a denumerable set. The points of non-denumerability of $\mathfrak{R}''$ are thus also points of non-denumerability of $\mathfrak{R}'''$. Therefore, $\mathfrak{R}_{nd}''$ is actually a subset of $\mathfrak{R}_{nd}'''$.

Let us denote by $\mathfrak{A} \equiv \{\alpha_1, \alpha_2, \dots\}$ the denumerable set of abscissas of the points of $\mathfrak{R}_d''$; and by $\mathfrak{B}$, the complementary set of abscissas of the points of $\mathfrak{R}_{nd}''$. Since $\mathfrak{R}_{nd}''$ is contained in $\mathfrak{R}_{nd}'''$, we have $s_f''(\xi) = s_f'''(\xi)$ for every ξ of $\mathfrak{B}$. Reasoning similar to that used above will show that $\mathfrak{R}_{nd}'''$ is contained in $\mathfrak{R}_{nd}^{IV}$, $\mathfrak{R}_{nd}^{IV}$ in $\mathfrak{R}_{nd}^V$, etc. Therefore, $\mathfrak{R}_{nd}''$ is contained in $\mathfrak{R}_{nd}^{(n)}$ for every finite $n > 2$; and hence $s_f''(\xi) = s_f^{(n)}(\xi)$ for every finite $n > 2$ and every ξ belonging to $\mathfrak{B}$. Accordingly, in virtue of our definition of $s_f^{(\nu)}(x)$ for transfinite ν (see Introduction), we conclude that $s_f''(\xi) = s_f^{(\omega)}(\xi)$, where ω is the first transfinite number. It is evident that the induction may be carried to every transfinite ordinal number of the second class.* Therefore, for every ξ of $\mathfrak{B}$ and every number $\nu > 2$ of the first or the second class, we have $s_f''(\xi) = s_f^{(\nu)}(\xi)$.

We thus see that the functions $s_f^{(\nu)}(x)$, where ν (≥ 2) ranges over all the numbers of the first or the second class, differ at most on the lines $x = \alpha_n$ $\{n = 1, 2, \dots\}$. We now show that these functions cannot all be different. For, if $s_f^{(\nu)}(\alpha_n) \neq s_f^{(\nu+1)}(\alpha_n)$, it follows from the fact that $s_f^{(\nu)}(x)$ is upper-semi-continuous, nowhere negative and equal to zero in an everywhere dense set, that $s_f^{(\nu+1)}(\alpha_n) < s_f^{(\nu)}(\alpha_n)$. Therefore, if $s_f^{(\nu+1)}(x) \neq s_f^{(\nu)}(x)$, at least one of the points $(\alpha_n, s_f^{(\nu)}(\alpha_n))$ of $\mathfrak{R}^{(\nu)}$ must "slide" a finite distance toward the X -axis in order to reach coincidence with the corresponding point $(\alpha_n, s_f^{(\nu+1)}(\alpha_n))$ of $\mathfrak{R}^{(\nu+1)}$. That is to say, to every pair of different consecutive functions $\{s_f^{(\nu)}(x), s_f^{(\nu+1)}(x)\}$, there corresponds at least one interval on the lines $x = \alpha_n$ through which the "sliding" just referred to takes place. It is evident, on account of the monotone decrease of $s_f^{(\nu)}(x)$ with increasing ν , that two intervals corresponding to two different pairs of different consecutive functions may abut but never overlap. Hence the number of such intervals on each of the lines $x = \alpha_n$ is at most denumerable; and therefore the total number of such possible intervals is at most denumerable. Consequently, there

* The theorem that the limit of a sequence of monotone decreasing upper-semi-continuous functions is upper-semi-continuous is here utilized in passing to limiting numbers, such as ω , of the second class. The reader will readily verify (see also the next paragraph) that the sequence

$$s_f''(x), s_f'''(x), \dots; s_f^{(\omega)}(x), s_f^{(\omega+1)}(x), \dots; \dots; \dots s_f^{(\nu)}(x), s_f^{(\nu+1)}(x), \dots; \dots$$

is monotone decreasing; the theorem just referred to applies to subsequences of this sequence that have the ordinal type of the set of positive integers arranged in ascending magnitude.

cannot be more than a denumerable number of different pairs of different consecutive functions $s_f^{(\nu)}(x)$. If now the functions $s_f^{(\nu)}(x)$ were all different for all numbers $\nu \geq 2$ of the first or the second class, there would be, contrary to the conclusion just reached, a non-denumerable infinity of pairs of different consecutive functions. Hence there must exist a number β of the first or the second class, such that $s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x)$, from which it follows that $s_f^{(\beta)}(x) \equiv s_f^{(\nu)}(x)$ for every $\nu > \beta$. Our proof is now complete.

We have shown above that if ξ belongs to $\mathfrak{B}$, then $s_f''(\xi) = s_f^{(\nu)}(\xi)$ for every $\nu > 2$ of the first or the second class. The converse also holds; namely, if $s_f''(\xi) = s_f^{(\nu)}(\xi)$ for every $\nu > 2$ of the first or the second class, then ξ belongs to $\mathfrak{B}$. This converse is a special case ($\lambda = 2$) of the following proposition:

If $(\xi, s_f^{(\lambda)}(\xi))$ belongs to $\mathfrak{R}_d^{(\lambda)}$, where $\lambda \geq 2$ is a number of the first or the second class, then for some number μ of the first or the second class $> \lambda$, we have

$$s_f^{(\lambda)}(\xi) > s_f^{(\mu)}(\xi).$$

For suppose that, contrary to the conclusion of this proposition, it were true that $s_f^{(\nu)}(\xi) = s_f^{(\lambda)}(\xi)$ for every ν of the first or the second class $> \lambda$. It would then follow, in virtue of the upper-semi-continuity of $s_f^{(\nu)}(x)$ and its monotone decrease with increasing ν , that $(\xi, s_f^{(\nu)}(\xi))$ belonged to $\mathfrak{R}_d^{(\nu)}$ for every ν of the first or the second class $> \lambda$. This is, however, impossible, on account of the following theorem and the fact shown above that for some β of the first or the second class $s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x)$.

THEOREM. The relation $s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x)$ implies that every point $(\xi, s_f^{(\beta)}(\xi))$ belongs to $\mathfrak{R}_{nd}^{(\beta)}$.

Proof. Suppose that $\mathfrak{R}_d^{(\beta)}$ were not the null-set, and the point

$$P \equiv (\xi, s_f^{(\beta)}(\xi)) \equiv (\xi, \eta)$$

an element of it. Then, in virtue of the upper-semi-continuity of $s_f^{(\beta)}(x)$ and the fact that P belongs to $\mathfrak{R}_d^{(\beta)}$, it is possible to enclose P in a region

$$R: \xi - \delta \leq x \leq \xi + \delta, \quad y \geq \eta - \delta \quad (\delta > 0),$$

such that R contains in its interior and on its boundary no more than a denumerable set of points of $\mathfrak{R}^{(\beta)}$. Moreover, we may and we do assume, that there is no point of $\mathfrak{R}^{(\beta)}$ on the boundary of R ; if originally R possesses such points, we may evidently select from the continuum of similar regions

$$R': \xi - \delta' \leq x \leq \xi + \delta', \quad y \geq \eta - \delta' \quad (\delta > \delta' > 0)$$

one having on its boundary none of the denumerable points of $\mathfrak{R}^{(\beta)}$ situated in R . Let $\mathfrak{C}$ represent the set of abscissas of points of $\mathfrak{R}^{(\beta)}$ that lie in R . I say that $\mathfrak{C}$ is closed. For suppose that $\{\gamma_n\}$, $n = 1, 2, \dots$, is a sequence of elements of $\mathfrak{C}$ having γ as a limit. Then the points $(\gamma_n, s_f^{(\beta)}(\gamma_n))$ are all in R . We wish to show that $(\gamma, s_f^{(\beta)}(\gamma))$ is also in R . From the sequence of points $(\gamma_n, s_f^{(\beta)}(\gamma_n))$, a subsequence $(\gamma_{m_n}, s_f^{(\beta)}(\gamma_{m_n}))$ may be extracted having as limit a single point (γ, k) , say, lying in the interior or on the boundary of R . Since $s_f^{(\beta)}(x)$ is upper-semi-continuous, it must be that $s_f^{(\beta)}(\gamma) \geq k$. Therefore, $(\gamma, s_f^{(\beta)}(\gamma))$ is not below (γ, k) , and hence, in R . $\mathfrak{C}$ is thus closed. Being, furthermore, denumerable, it must be reducible, and it therefore contains at least one isolated value, say $\bar{\gamma}$. A fortiori, the point $(\bar{\gamma}, s_f^{(\beta)}(\bar{\gamma}))$, being an interior point of R , is an isolated point of $\mathfrak{R}^{(\beta)}$; and therefore, because of the upper-semi-continuity of $s_f^{(\beta)}(x)$ and the everywhere dense distribution of its zeros, we have

$$s_f^{(\beta+1)}(\bar{\gamma}) < s_f^{(\beta)}(\bar{\gamma}).$$

Therefore, if $\mathfrak{R}_d^{(\beta)}$ is not the null-set, $s_f^{(\beta+1)}(x) \not\equiv s_f^{(\beta)}(x)$.

From the above considerations, we may deduce the following interesting connection between the d -saltus and the f -saltus.

THEOREM. *For every function $g(x)$, for which $s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x)$, we have*

$$s_d(s_f'', x) \equiv s_f^{(\beta)}(x).$$

For the proof, further details of which are left to the reader, we note that $(\xi, s_f^{(\beta)}(\xi))$ is the highest point of non-denumerability of $\mathfrak{R}''$ —of course, not necessarily belonging to $\mathfrak{R}''$ —on the line $x = \xi$.

5. The e -saltus. **THEOREM.** $s_e'''(x) \equiv 0$.

Proof. Just as in the previous two cases, it is seen that $s_e''(x) = 0$ at the points of continuity of $s_e'(x)$. Since the points of discontinuity of $s_e'(x)$ constitute an exhaustible set $\mathfrak{S}$,* we have $s_e''(x) = 0$ everywhere except possibly in $\mathfrak{S}$. The values of $s_e''(x)$ in $\mathfrak{S}$ may be neglected when $s_e'''(x)$ is calculated. Hence $s_e'''(x) = 0$.

The following example shows that $s_e''(x)$ need not equal $s_e'''(x)$. Let $(a, b) \equiv (0, 1)$, and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two complementary sets of the interval $(0, \frac{1}{2})$, such that both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are inexhaustible (i. e., of second category) in every subinterval of $(0, \frac{1}{2})$.† Let $g(x) = 0$ if x belongs to $\mathfrak{M}_1$, and $g(x) = 1$ if x does not belong to $\mathfrak{M}_1$. Then

* See, for example, Hobson, l. c., p. 244.

† For the existence of such complementary sets, see the example of Mahlo in Schoenflies-Hahn, *Entwicklung der Mengenlehre und ihrer Anwendungen* (1913), p. 349. The example is on Zermelo's "*Auswahlpostulat*."

$$s_e'(x) = 1 \quad (0 \leq x \leq \tfrac{1}{2}), \quad s_e'(x) = 0 \quad (\tfrac{1}{2} < x \leq 1);$$

$$s_e''(x) = 0 \quad (x \neq \tfrac{1}{2}), \quad s_e''(\tfrac{1}{2}) = 1; \quad s_e'''(x) \equiv 0.$$

Remark. If $s_t'(x)$, $s_t''(x)$, $s_t'''(x)$, $\dots$ represent the successive saltus functions that arise when, instead of neglecting exhaustible sets, we neglect nowhere-dense sets, then $s_t'''(x) \equiv 0$. For, as in the case of the e -saltus, we show that $s_t''(x)$ is 0 everywhere except in an exhaustible set. Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$; and let $\mathfrak{C}_n$ be the set of abscissas x at which $s_t''(x) \geq \epsilon_n$. Then $\mathfrak{C}_n$ is a nowhere-dense set.* $\mathfrak{C}_n$ may therefore be neglected in the calculation of $s_t'''(x)$. For the remaining points of (a, b) , we have $s_t''(x) < \epsilon_n$. This inequality, holding for every n , shows that $s_t'''(x) \equiv 0$.

6. The z -saltus. THEOREM. $s_z'''(x) \equiv s_z^{IV}(x) \equiv s_z^V(x) \equiv \dots$

The proof proceeds along the same lines as that of the corresponding theorem of § 3.† We shall, however, make clear how the parallelism is effected.

As in the previous cases, we see that $s_z''(x)$ is a nowhere negative, upper-semi-continuous function, whose zeros are everywhere dense in (a, b) . Let (α, β) be any subinterval of (a, b) and γ a point in the interior of (α, β) for which $s_z''(\gamma) = 0$. Since $s_z''(x)$ is upper-semi-continuous, there exists, for every $\epsilon > 0$, a subinterval $(\gamma - \delta, \gamma + \delta)$ of (α, β) throughout which $s_z''(x) < \epsilon$. Therefore, no matter what set of zero measure is neglected, there still remain points in $(\gamma - \delta, \gamma + \delta)$ at which $s_z''(x) < \epsilon$. This shows that $l_z(s_z'', \alpha\beta) < \epsilon$ for every subinterval (α, β) and every $\epsilon > 0$. Therefore, for every x ,

$$l_z(s_z'', x) = 0,$$

and

$$s_z'''(x) = u_z(s_z'', x) - l_z(s_z'', x) = u_z(s_z'', x) \leq s_z''(x).$$

Now let $\mathfrak{R}''$ be the planar set of points whose coördinates are $x = x$, $y = s_z''(x)$ ($a \leq x \leq b$). If P is any point of the plane, whether belonging to $\mathfrak{R}''$ or not, there are two mutually exclusive possibilities: (1) There exists a neighborhood of P such that the projection upon the X -axis of the set of points of $\mathfrak{R}''$ in it is a set of measure 0. (2) Every neighborhood of P contains a set of points of $\mathfrak{R}''$ whose projection upon the X -axis is not a set of measure 0.‡ In the first case, we shall say that P is a “ z -point” of $\mathfrak{R}''$; in the second case, that P is a “ $\bar{z}$ -point” of $\mathfrak{R}''$.

* Cf., for example, Hobson, l. c., pp. 243 and 244.

† It would be easy to set up, on the basis of a suitable system of postulates, a theory having as consequence a common generalization of the theorem of the present section and of that in § 3. Cf. E. H. Moore, *Introduction to a Form of General Analysis* (1910), p. 1. Cf. also Appendix.

‡ The projection is then either of measure > 0 or non-measurable.

Just as in the case of the d -saltus, we may now show that for every ξ of (a, b) , the point $(\xi, s_z'''(\xi))$ is the highest $\bar{z}$ -point of $\mathfrak{R}''$ on the line $x = \xi$. All we have to do is to substitute, in the proof of the corresponding statement of § 3, “ z -point” for “point of denumerability,” “ $\bar{z}$ -point” for “point of non-denumerability,” “set of points whose projection on the X -axis is of zero measure” for “denumerable set,” s_z'' for s_d'' , etc. In particular, it follows, as there, that if $(\xi, s_z''(\xi))$ is itself a $\bar{z}$ -point of $\mathfrak{R}''$, then $s_z'''(\xi) = s_z''(\xi)$.

To complete the proof in the analogous fashion intended, it will now be seen that all we need is the analogue of the theorem in § 3, that the points of denumerability of $\mathfrak{R}''$ constitute at most a denumerable set; namely, the theorem, that the projection upon the X -axis of the set of z -points of $\mathfrak{R}''$ is a set of zero measure. This may be easily proved as follows. Divide the plane into squares by means of the lines $x = \pm n/2^k$, $y = \pm n/2^k$, where $n = 0, 1, 2, \dots$, and k is a positive integer. For a fixed k , we get a single system of squares. Letting $k = 1, 2, 3, \dots$, we obtain a denumerable set of systems of squares. Let us call the square S a “ z -square,” if it belongs to any one of these systems and is furthermore such that the projection upon the X -axis of the set of points of $\mathfrak{R}''$ in the interior or on the boundary of S is a set of zero measure. Since the number of “ z -squares” is denumerable, the projection upon the X -axis of the totality of points of $\mathfrak{R}''$ in the interior or on the boundary of all the “ z -squares” must be a set of zero measure; for the measure of a set of denumerable sets is the sum of the measures of these sets. But, as may be readily seen from the definition of “ z -point” and of “ z -square,” every z -point is in the interior or on the boundary of at least one z -square. Hence, the projection of the totality of z -points of $\mathfrak{R}''$ is of zero measure, and our theorem is proved.

The following example shows that $s_z''(x)$ need not equal $s_z'''(x)$. Let $(a, b) \equiv (0, 1)$ and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two complementary sets of the interval $(0, \frac{1}{2})$ such that in every subinterval of $(0, \frac{1}{2})$ both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are either non-measurable or of measure > 0 .* Let $g(x) = 0$, if x belongs to $\mathfrak{M}_1$, and $g(x) = 1$, if x does not belong to $\mathfrak{M}_1$. Then

$$\begin{aligned} s_z'(x) &= 1 & (0 \leq x \leq \tfrac{1}{2}), & & s_z'(x) &= 0 & (\tfrac{1}{2} < x \leq 1); \\ s_z''(x) &= 0 & (x \neq \tfrac{1}{2}), & & s_z''(\tfrac{1}{2}) &= 1; & & s_z'''(x) \equiv 0. \end{aligned}$$

Remark. An apparently different type of saltus may be defined as follows. If (α, β) is any subinterval of (a, b) , let $s_e(g, \alpha\beta)$ be the saltus

* For the existence of such complementary sets, see Schoenflies-Hahn, l. c., pp. 374-378, where examples showing the required existence, constructed by Vitali, Lebesgue, Van Vleck and Hausdorff, are described. All these examples are based on Zermelo's “Auswahlpostulat.”

of $g(x)$ in (α, β) , if sets of measure ϵ —or of measure $\epsilon/(\beta - \alpha)$ —may be neglected. (Cf. the definitions given in the Introduction.) As ϵ decreases, $s_\epsilon(g, \alpha\beta)$ surely does not decrease; and therefore, since $g(x)$ is bounded, $s_\epsilon(g, \alpha\beta)$ has a definite limit for $\epsilon \rightarrow 0$, which we may denote by $s_{\epsilon \rightarrow 0}(g, \alpha\beta)$. It is readily seen, however, that $s_{\epsilon \rightarrow 0}(g, \alpha\beta)$ is identical with our $s_z(g, \alpha\beta)$.

Appendix—Remarks on Generalizations.

Throughout our discussion so far, we have dealt with a real function $g(x)$, defined in an interval (a, b) and further restricted to being single-valued and bounded. In fact, however, our results are essentially far more general. It is not difficult to see how most of our definitions and proofs may be extended to the case of many-valued, bounded or unbounded functions of several variables, or of infinitely many variables. But even such an extension would be short of the full intrinsic generality of our considerations. The reader cannot have failed to notice that throughout the discussion, only a small number of basal notions and properties were utilized. To indicate, if but vaguely, the degree to which the generalization of some of our results may be carried, we refer to a paper, read by the author before the American Mathematical Society.* In this paper, it was shown how several of the theorems in the present paper, and others closely related, may be proved for a “neighborhood” theory of a class $\mathfrak{R}$, in which the main assumption—there are other postulates playing a minor rôle—is approximately as follows: If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are both neighborhoods of the element P of $\mathfrak{R}$, then there exists a neighborhood of P contained in both $\mathfrak{N}_1$ and $\mathfrak{N}_2$. The publication of the suggested generalization of our present results is, however, reserved for the future.

* See notice in Bulletin of the American Mathematical Society, March, 1915, p. 289.